Math 246B Lecture 15 Notes

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1 Corollaries of Weierstrass's Theorem and Entire Functions of Finite Order

1.1 Existence of a holomorphic function with given Taylor expansion near infinitely many points

Last time, we proved Weierstrass's theorem, which says that if $A \subseteq \Omega$ is a set with no limit points, then we can construct $f \in \mathrm{Hol}(\Omega)$ with zero set A (with multiplicities).

Proposition 1.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $A = \{\alpha_j\}_{j=1}^\infty$ be an infinte set with no limit points in Ω. For each $j \geq 1$, let $m_j \geq 0$ be an integer, and let f_j be holomorphic near α_j. Then there exists some $f \in \mathrm{Hol}(\Omega)$ such that for all j, $f(z) - f_j(z)$ is $O(|z - \alpha_j|^{m_j+1})$ as $z \to \alpha_j$. (Thus, the Taylor expansion of f can be prescribed up to order m at each α_j.)*

Proof. By Weierstrass's theorem, we can construct $g \in \mathrm{Hol}(\Omega)$ have zeros of order $m_j + 1$ at α_j for all j. By Mittag-Leffler's theorem, there exists a meromorphic function h in Ω with poles at $\{\alpha_j\}$ only such that $h - f_j/g = r_j$ is holomorphic near α_j for all j. Define $f = gh \in \mathrm{Hol}(\Omega \setminus A)$. Then $f/g - f_j/g$ is holomorphic near α_j, so $f - f_j$ is holomorphic near α_j. So $f \in \mathrm{Hol}(\Omega)$. Also, $f - f_j = r_j g$, where r_j is $O(1)$ and g is $O(|z - a_j|^{m_j+1})$ as $z \to \alpha_j$. □

1.2 Existence of a holomorphic function which cannot be extended

Here is another corollary of Weierstrass's theorem.

Corollary 1.1. *Let Ω be open. There exists $f \in \mathrm{Hol}(\Omega)$ which cannot be continued analytically to any larger open set. More precisely, if $a \in \Omega$, $g \in \mathrm{Hol}(D(a, r))$, and $f = g$ near a, then $D(a, r) \subseteq \Omega$.*

We say that Ω is the **natural domain of holomorphy** for f.

Proof. Let $\{\alpha_k\}_{k=1}^\infty$ be an enumeration of all points in Ω with rational coordinates. Let $(z_j)_{j=1}^\infty$ be a sequence in Ω such that each α_k such that each α_k occurs an infinite number of times: $(\alpha_1, \alpha_1, \alpha_1, \alpha_2, \alpha_1, \alpha_2, \alpha_3, \alpha_1, \dots)$. Choose a compact exhaustion of Ω: $K_j \subseteq \Omega$ with $K_j \subseteq K_{j+1}^o$ and $\bigcup_j K_j = \Omega$. Let $r_j = \operatorname{dist}(z_j, \mathbb{C} \setminus \Omega)$ so that $D(z_j, r_j)$ is the largest open disc centered at z_j contained in Ω. For each j, let $w_j \in D(z_j, r_j) \setminus K_j$. We let $A = \{w_j\}$; each compact set is contained in K_j for some j, so A has no limit points in Ω. Thus there exists $f \in \operatorname{Hol}(\Omega)$ such that $f^{-1}(\{0\}) = A$. Now let $a \in \Omega$ have rational coordinates and consider $D(a, r)$, where $r = \operatorname{dist}(a, \mathbb{C} \setminus \Omega)$. We have: $a = z_j$ for infinitely many j, so $D(a, r)$ contains infinitely many points w_j. Thus, by the uniqueness of analytic continuation, no function which is equal to f near a can be holomorphic in any larger disc centered at a. $\square$

Remark 1.1. When $n > 1$, this property does not hold for functions in $\mathbb{C}^n$.

1.3 Entire functions of finite order

Definition 1.1. We that $f \in \operatorname{Hom}(\mathbb{C})$ is of **finite order** if there is some $\sigma \in \mathbb{R}$ such that $|f(z)| \leq Ce^{|z|^\sigma}$ for all $z \in \mathbb{C}$ for some $C > 0$. The **order** ρ of f is the infimum of such σ.

Observe that $\rho \in [0, \infty)$. Also, f has order ρ iff for all $\varepsilon > 0$, $f(z)/e^{|z|^{\rho+\varepsilon}}$ is bounded on $\mathbb{C}$ and $f(z)/e^{|z|^{\rho-\varepsilon}}$ is unbounded on $\mathbb{C}$.

Example 1.1. Polynomials have order 0.

Example 1.2. e^z, $\cos(z)$, and $\sin(z)$ all have order 1. The function ze^z still has order 1. The function e^{z^m} has order m.

Example 1.3. The order need not be an integer. For example, $\cos(\sqrt{z})$ (defined by its Taylor expansion) has order $1/2$.

Example 1.4. Let $f \in L^1(\mathbb{R})$ be compactly supported; that is, there exists some R such that $f(x) = 0$ for a.e. x with $|x| > R$. Then the **Fourier transform** of f,

$$\hat{f}(\xi) = \int e^{-ix\xi} f(x)\, dx$$

for $\xi \in \mathbb{R}$, can be extended to the entire function

$$\hat{f}(\zeta) = \int e^{-ix\zeta} f(x)\, dx$$

for $\zeta \in \mathbb{C}$. Then

$$|\hat{f}(\zeta)| \leq \int_{-R}^{R} e^{x \operatorname{Im}(\zeta)} |f(x)|\, dx \leq e^{R|\zeta|} \|f\|_{L^1},$$

so $\hat{f}$ is of order ≤ 1.

Remark 1.2. Let $M(r) = \max_{|z|=r} |f(z)|$. We have

$$\rho = \limsup_{r \to \infty} \frac{\log(\log(M(r)))}{\log(r)} = \lim_{R \to \infty} \left(\sup_{r \geq R} \frac{\log(\log(M(r)))}{\log(r)} \right).$$